TOPICS IN PROBABILITY THEORY AND STOCHASTIC PROCESSES
Home Works 10 & THE FINALE due on THURSDAY December 11,
Instructor: Prof. Artem Zvavitch

Problem 1. *Consider the experiment of rolling two dice. Let X be the value of the first roll and Y the sum of the two dice. Find $\mathbb{E}(X|Y)$ (i.e. give the value of $\mathbb{E}(X|Y=y)$ for all y.*

Problem 2. *Suppose that X_t is a Poisson process with parameter $\lambda = 1$. Find $\mathbb{E}X_1|X_2$ and $\mathbb{E}X_2|X_1$.*

Problem 3. *Let $X_1, X_2, \dots$ be independent Identically distributed random variables. Le $\phi(t) = \mathbb{E}e^{tX_1}$ be the moment generating function of X_1 (and thus of all X_i). Fix t and assume $\phi(t) < \infty$. Let $S_0 = 0$ and for $n > 0$ let*

$$S_n = X_1 + \cdots + X_n.$$

Let $M_n = \phi(t)^{-n}e^{tS_n}$ Show that M_n is a martingale with respect to $X_1, X_2, \dots$.

Problem 4. *Let $X_0, X_1, \dots$ be the values of a branching process, i.e. X_n gives the number of individuals in the nth generation. Suppose that each individual produce offspring from distribution with mean μ and variance δ^2.*

- *Show that $M_n = \mu^{-n}X_n$ is a martingale with respect to $X_1, X_2, \dots$.*
- *Let F_n denote the information contained in $X_0, X_1, \dots, X_n$. Show that*
  $$\mathbb{E}(X_{n+1}^2|F_n) = \mu^2 X_n^2 + \delta^2 X_n.$$
- *Suppose $\mu > 1$. Show that there exists a constant $C < \infty$ such that for all n*
  $$\mathbb{E}M_n < C$$
- *Show that this is not the case for $\mu \leq 1$.*

Problem 5. *Suppose that N is a standard normal random variable and $X = ae^{bN}$, where $a, b >$. Show that the density X is*

$$f(x) = \frac{1}{xb}\frac{1}{2\pi}e^{-\frac{(\log(x/a))^2}{2b^2}}.$$

Also show that if $K > 0$, then

$$\int_0^\infty (x-K)^+ f(x)dx = ae^{b^2/2}\Phi\left(\frac{\log(a/K)+b^2}{b}\right) - K\Phi\left(\frac{\log(a/K)}{b}\right),$$

where Φ denotes the distribution function of N.