

**INTRODUCTION TO TOPOLOGY I**  
**Home Work 4, due on Tuesday, October 7.**  
**Instructor: Prof. Artem Zvavitch**

**Problem 1.** Consider the product, uniform and the box topologies on  $\mathbb{R}^\omega$ . In which topologies are the following functions from  $\mathbb{R} \rightarrow \mathbb{R}^\omega$  continuous?

- $f(t) = (t, 2t, 3t, \dots)$ .
- $g(t) = (t, t, t, t, \dots)$ .
- $h(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \dots)$ .

**Problem 2.** In  $\mathbb{R}^n$  define  $d_1(x, y) = \|x - y\|_1 = \sum_{i=1}^n |x_i - y_i|$

- Show that this is a metric.
- Prove that this induces the usual topology in  $\mathbb{R}^n$ .
- Consider the  $X \subset \mathbb{R}^\omega$ , where  $X$  is set of all sequences so that  $\sum |x_i|$  converges. On  $X$  we have three topologies it inherits from the box, uniform and product topologies on  $\mathbb{R}^\omega$ . We have also a topology given by metric  $d_1$ . Please compare those 4 topologies!

**Problem 3.** Show that  $\mathbb{R} \times \mathbb{R}$  in dictionary order topology is metrizable.

**Problem 4.** Prove that if  $X$  is a metric space with metric  $d$  then  $d : X \times X \rightarrow \mathbb{R}$  is a continuous function.

**Problem 5.** Show that if  $d$  is a metric on  $X$ , the

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is bounded metric that gives the topology of  $X$ .

**Problem 6.** define  $f_n : [0, 1] \rightarrow \mathbb{R}$  as  $f_n(x) = x^n$ . Show that the sequence  $(f_n(x))$  converges for  $x \in [0, 1]$ , but does not converge uniformly.